YITP-00-41 February 1, 2008

The Supersymmetric Kink via Higher-Derivative and Momentum Cut-Off Regularization Schemes

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Abstract

We study the supersymmetric kink with higher derivative and momentum cut-off regularization schemes. We introduce the new momentum cut-off regularization scheme which we call "generalized momentum cut-off". A new, explicit computation for the central charge anomaly for this scheme is described in detail. The calculation of the first order mass corrections for the bosonic and supersymmetric kink within the momentum cut-off is presented in [1], so that one can compare the one-loop central charge and mass computed independently within the same regularization setup. We confirm that the BPS bound is saturated in one loop level. Thus the Wilsonian momentum cut-off regularization scheme is rehabilitated as a bona fide procedure for computing quantum corrections in the topologically nontrivial backgrounds. We lead the reader to the idea that a consistent regularization in general does not only assume that one regulates loops properly, but also requires caution in defining the total number of modes involved in the quantization of the theory. We also study the higher-derivative regularization scheme of [2] in great detail. We extend the Noether method for the case when the higher-derivatives are present in the Lagrangian. The extensive discussion of the technical aspects and the consistency of the higher-derivative scheme is given. We show that higher-derivative regularization gives, in general, a nonlocal topological current, which leads to the correct value of the anomaly in the central charge. We also comment on the status of the dimensional regularization approach to the computation of the central charge anomaly and kink mass.

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1 Introduction

The problem of the quantum corrections to the mass of the (1+1) dimensional supersymmetric kink - the topologically nontrivial configuration in a system with the Lagrangian

$$L = \frac{1}{2}\left\{\partial_\mu\phi\partial^\mu\phi + \bar{\psi}i\,\partial\!\!\!/\psi + F^2 + 2W'(\phi)F - W''(\phi)\bar{\psi}\psi\right\}, \tag{1}$$

with

$$W'(\phi) = \frac{m^2}{4\lambda} - \lambda\phi^2 \tag{2}$$

has had more than twenty years of history. Different perturbative computations gave different answers and only recently most of the loose ends of different approaches were tide up to form the consistent picture. A review of this picture was given in [1].

The mass of the bosonic or supersymmetric kink is defined as being the difference between the vacuum expectation values (VEVs) of Hamiltonian in the nontrivial and trivial backgrounds. Both of these VEVs are divergent and the difference is meaningless unless one uses some regularization scheme to make these sums finite (or at least logarithmically divergent). A general expression for the mass is clear from the following scaling property: under the rescaling

$$\breve{z} = \Lambda^{-1/2}z, \breve{t} = \Lambda^{-1/2}t, \breve{\phi} = \Lambda^{1/2}\phi, \breve{\psi} = \Lambda^{3/4}\psi, \breve{\hbar} = \Lambda\hbar, \breve{m} = \Lambda^{1/2}m$$

the VEV of Hamiltonian scales as

$$\langle H(m,\lambda)\rangle_\hbar = \frac{1}{\Lambda^{3/2}}\langle H(\Lambda^{1/2}m,\lambda)\rangle_{\Lambda\hbar}\ ,$$

which implies (we also use that both λ and m in (2) both have the dimensions of mass) the following form of this VEV

$$\langle H(m,\lambda)\rangle_\hbar = a\frac{m^3}{\lambda^2} + b\hbar m + c\hbar^2\frac{\lambda^2}{m} + O(\hbar^3)$$

where a,b,c are some numerical coefficients. For the last twenty five years, there were a lot of controversal computations of the value of b in the literature (see [3] for further references). The agreement on this value was reached only recently after the introduction of the derivative regularization scheme in ref. [4]. The generally accepted values of b for bosonic and susy kinks are

$$b(bos) = -m\left(\frac{3}{2\pi} - \frac{\sqrt{3}}{12}\right); \qquad b(susy) = Z^{(1)}(susy) = -\frac{m}{2\pi} \tag{3}$$

Once the mass value was settled, the next important question, the BPS bound saturation, was addressed by [4]. Though mentioning the word “anomaly”, [4] failed to check the saturation. The real understanding came with [2], where it was realized that there is the quantum ultraviolet anomaly in the central charge which is responsible for the exact saturation of the BPS bound. In our opinion, the pioniering work [2] had some technical mistreatments in the implementation of the higher derivative and dimensional regularization

schemes. It is the goal of the present work to clarify the issue of the anomaly in the central charge by, at one hand, giving a new computation of the anomaly in the new generalized momentum cut-off scheme, and, on the other hand, by presenting a different, self-consistent computation of an anomaly with the higher-derivative scheme of ref. [2].

We start with the momentum cut-off regularization scheme. This is one of the most popular schemes for the perturbative calculations. Nevertheless, previous, naive implementations of this scheme did not disclose the anomaly in the central charge. We solve this problem. We notice that it is important to regularize the problem consistently *before* we start to do any computations. This is a general principle that must be used in any regularization procedure: regularization scheme is a part of the model and should be put in it from the very start. For the case of momentum cut-off scheme, this rule says that one should declare that only plane waves with momenta up to cut-off are considered in the theory, put into path integrals and used to expand quantum fields in second quantized form. In particular, the propagators and the Dirac delta functions in the canonical commutation relations should contain only the modes with momenta smaller than a cut-off, that is we must impose a cut-off on Dirac delta functions as well as on loops. Of course, this cut-off should be exactly the same as in the propagators. Nevertheless, we first put two different cut-offs, one is for quantum fields, another is for delta functions. (This will allow us to understand easily some results of the higher-derivative regularization approach). We observe that there are only three different results for the anomaly that are possible, depending on which cut-off is larger, or both are equal. For equal cut-offs we reproduce the correct value of the anomaly.

We would like also to point out that in paper [1] we develop an accurate momentum cut-off treatment of the Casimir energy sum, thus computing directly the mass of the supersymmetric kink. To the best of our knowledge this is the first time when the anomaly in the central charge and the quantum mass are computed independently within the framework of the same regularization setup. In our case, this setup is the momentum cut-off scheme [2].

Another scheme, the higher-derivatives regularization, was proposed by [2]. All the inputs of this scheme are written in the superspace Lagrangian, thus the supersymmetry is manifestly preserved by this regularization. We study this regularization scheme in great detail. We develop the Noether method for higher-derivative Lagrangian. The resulting supersymmetry current is not unique. Using Noether method with susy tranformations one can compute a whole family of currents j^{μ}, which differ from each other by a trivially conserved term of the form $\epsilon^{\mu\nu}\partial_{\nu}A(x,t)$. Notice that, in general, not only fermionic anticommutators contribute to the central charge, but also bosonic canonical commutation relations give a finite first-order contribution. These two terms always add to give a proper value for the anomaly. The fact that the above mentioned nonuniqueness could not possibly change the central charge value also follows from the general form of the susy algebra, as it was pointed to us by [5]. Nevertheless, we claim that only one of the currents presented in this paper can be used for the anomaly computation. The reason is that the other currents are not properly

[2] We are aware of paper [6] which claims to do the same. We will describe later in this paper why one should understand [6] as only being valid in the context of the dimensional regularization. Also, that paper does not refer to the central charge anomaly, and our understanding is that it just reduces the computation of central charge VEV to the computation of the hamiltonian VEV plus zero, arising from the VEV of the susy generator component squared.

regulated by the theory, vacuum averages of this currents are divergent, though the space integration leaves only finite contributions to the anomaly. One must pick some very special current which we call j^{μ}_{OK} to have the loops regulated properly. This current is different from the one used in [2]. We believe that we present an accurate and consistent treatment of the model and we compute the correct anomaly value from the properly regulated current.

The correct result for the anomaly turns out to be exactly equal to the one loop mass correction, computed by [4], and then checked by different methods in [6] and [7]. That is, the BPS bound is saturated in one loop, which confirms the claim made in [1, 8] (after reanalysing arguments of [2]) that there is multiplet shortening in $1+1$ dimensions and the BPS bound is saturated in any number of loops.

2 Supersymmetric kink in momentum cut-off regularization

We begin with some definitions[3]. Our metric is $(+-)$, $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$, and $\bar{\psi} = \psi^T \gamma^0$. The action for a scalar superfield Φ is

$$S = i \int d^2\theta d^2x \left\{ \frac{1}{4} \bar{D}^{\alpha} \Phi D_{\alpha} \Phi + W(\Phi) \right\} \tag{4}$$

where we will take $W(\Phi) = \frac{m^2}{4\lambda}\Phi - \frac{\lambda}{3}\Phi^3$. The corresponding Lagrangian in component fields is given by (1). The action is invariant under the supersymmetry transformations given by

$$\begin{aligned} \delta\phi &= \bar{\epsilon}\psi \\ \delta\psi &= -i(\partial\!\!\!/\phi)\epsilon + F\epsilon \\ \delta F &= -i\bar{\epsilon}(\partial\!\!\!/\psi) \end{aligned} \tag{5}$$

The nontrivial classical solution of the corresponding equations of motion is the kink

$$\phi_0(x) = \frac{m}{2\lambda} \tanh \frac{mx}{2} \tag{6}$$

To study the quantum theory with the action (4) one needs to regulate and renormalize. Only mass renormalization is required for this model. We will adopt the most popular renormalization scheme, which requires that the one-loop tadpole graphs are exactly cancelled by the mass counterterm, see ref.[1] for details. In this chapter we pick the momentum cut-off regularization scheme to do that, i.e. in the quantum fields we are going to keep only modes with the momenta smaller than the cut-off scale Λ. Note that, in general, this implies that the classical solution (6) is no longer correct - one needs to remove from it the modes with the momenta higher than cut-off Λ. Nevertheless, this won't influence further discussion because such a correction would vanish when the cut-off is pushed to infinity. (This will be

[3] Note that our notations are the same as in ref. [2] and they differ from the conventions of ref. [1].

clear from the following discussion and eq. (12) if one keeps in mind that only the antisymmetric modes $\sin kx$ contribute to the Fourier expansion of (6)). We impose the equal time commutation relations

$$[\phi(t,x), \partial_0\phi(t,x')] = i\hat{\delta}(x-x') \qquad \left\{\psi_\alpha(t,x), \psi^{T\beta}(t,x')\right\} = \delta_\alpha^\beta \hat{\delta}(x-x') \tag{7}$$

where we keep a finite number of Fourier modes, namely the modes with momenta below cut-off K in delta function, i.e.

$$\hat{\delta}(x) = \int_{-K}^{K} \frac{dq}{2\pi} \exp(iqx) \tag{8}$$

The theory is consistent only if $\Lambda = K$. Nevertheless, we wish to keep these two parameters different. The result in terms of both of these parameters will describe the nature of problems that one may encounter in much more complicated schemes.

Using Noether method, one can compute the supersymmetry current

$$j^\mu = (\partial_\nu\phi)\gamma^\nu\gamma^\mu\psi + iW'\gamma^\mu\psi \tag{9}$$

The supersymmetry charge Q is defined as the space integral of the time component of this current. We are computing the supercharge algebra

$$\{Q_\alpha, \bar{Q}^\beta\} = 2(\gamma_\nu)_\alpha^\beta P^\nu + 2i(\gamma^5)_\alpha^\beta Z \tag{10}$$

where we have introduced the central charge Z and the momentum P^ν. We are going to compute the VEV of Z in one loop. Consider the anticommutator in (10). From counting the number of γ-matrices it is clear that the central charge contributions come from two sources:

1) from the fermionic anticommutator of fermionic field of the first term in the current with the fermionic field of the second term in the current, and then computing bosonic loop;

2) from the bosonic commutator of $\partial_0\phi$ term in the first term of the current with the bosonic field in the second term of the current.

For the case 1) one computes

$$Z_{bos} = \int_{-L}^{L} dx \int_{-L}^{L} dx' W'(x)\partial_{x'}\phi(x')\hat{\delta}(x-x') \tag{11}$$

This is an operator expression. (We suppress the time dependence of fields for the compactness of formulas). We now write the field as $\phi(x) = \phi_0(x) + \eta(x)$ and compute the VEV of this operator in one loop. This gives

$$\langle Z_{bos}\rangle = Z_0 + \int_{-L}^{L} dx \int_{-L}^{L} dx' W'''(x)\partial_{x'}\phi_0(x')\hat{\delta}(x-x')\langle\eta^2(x)\rangle +$$

$$\int_{-L}^{L} dx \int_{-L}^{L} dx' W''(x)\langle\eta(x)\partial_{x'}\eta(x')\rangle\hat{\delta}(x-x') \tag{12}$$

where Z_0 is the classical term, and the rest is the one loop correction. The second term in this expression can be written as a total derivative of W'', and we recognize the regular

logarithmic term which comes from the second order Taylor expansion when one computes the one loop correction to $Z = W$ term. This term can be removed by the renormalization.

The third term in (12) is responsible for the half of the total anomaly (another half being given by the case 2)). The correlator in this term is an odd function of $(z - z')$. So, we need to compute the integral

$$\int_{-L}^{L} dx \int_{-L}^{L} dx' W''(x) i \int_{-\Lambda}^{\Lambda} \frac{d^2p}{(2\pi)^2} \frac{ip_x}{p^2} \exp(ip_x(x - x')) \int_{-K}^{K} \frac{dq}{2\pi} \exp(iq(x - x')) \tag{13}$$

We find for this integral in the limit $L \to \infty$

$$\frac{W''(\infty)}{4\pi} \left(1 + \operatorname{sign}(\Lambda - K)\right) \tag{14}$$

Of course, it is easy to compute this integral by computer. We feel that the reader, who likes to have all the computations under the control, may have some technical problems computing this integral and that is why we devote the appendix A to this computation.

Now, let us move to the case 2). Consider what one would get from commutators of $\partial_0 \phi$ with ϕ and $\partial_x \phi$ in the supersymmetry charge. The γ^5 part results in the following integral for the central charge contribution

$$\int_{-L}^{L} dx \int_{-L}^{L} dx' \left\{ i\partial_{x'} \hat{\delta}(x - x') \langle \bar{\psi}(x') \psi(x) \rangle + W''(x) \hat{\delta}(x - x') \langle \bar{\psi}(x') \gamma^1 \psi(x) \rangle \right\}$$

The first term here is zero (the p term in the fermionic propagator is killed by trace of gamma matrix, the m term is just zero due to the parity properties). The γ^1 in the second term picks the x component of the loop momentum and we get exactly the same mathematical expression for the second term as one in the (13).

This way we observe that the total result for the central charge anomaly is

$$\frac{W''(\infty)}{2\pi} \left(1 + \operatorname{sign}(\Lambda - K)\right) \tag{15}$$

and bosonic and fermionic loops contribute into it equally.

The formula (15) has interesting properties. What if we used the full delta function instead of $\hat{\delta}$? Then it would remove one of the coordinate integrals from the very start, and we would loose the anomaly! This is generally true for any $K > \Lambda$, the anomaly computed in such a scheme is zero. This is the reason why this anomaly has been overlooked by earlier papers. On the other hand, if we would under-regulate propagators, i.e. put $\Lambda > K$, then the anomaly which is computed in this way would be twice the one, that we present here. It is needless to say that both cases result in the internally inconsistent theory.

Thus we conclude that for properly regularized theory, i.e. for $\Lambda = K$, the central charge anomaly is

$$\frac{W''(\infty)}{2\pi}$$

and the BPS bound is saturated.

The scheme discussed here violates supersymmetry. It would be useful to repeat this analysis for the manifestly supesymmetric regularization. We are doing it in the next chapter for the regularization scheme, proposed by [2].

3 Supersymmetric kink in higher-derivative regularization

Consider the regularized action for a scalar superfield Φ in the superspace

$$S = i\int d^2\theta d^2x \left\{ \frac{1}{4}\bar{D}^\alpha\Phi\left(1 - \frac{\partial_z^2}{M^2}\right)D_\alpha\Phi + W(\Phi)\right\} \tag{16}$$

where M is a regulator mass. This scheme does not have a sharp cut-off scale Λ, as the model before, but it introduces the characteristic scale M at which the momentum loops are becoming suppressed. This is a smooth cut-off counterpart of the sharp momentum cut-off scheme. The corresponding Lagrangian in component fields is

$$\begin{aligned} L = \frac{1}{2}\Bigg\{ &\partial_\mu\phi\left(1 - \frac{\partial_x^2}{M^2}\right)\partial^\mu\phi + \bar{\psi}\left(1 - \frac{\partial_x^2}{M^2}\right)i\,\partial\!\!\!/\psi \\ &+F\left(1 - \frac{\partial_x^2}{M^2}\right)F + 2W'(\phi)F - W''(\phi)\bar{\psi}\psi\Bigg\} \end{aligned} \tag{17}$$

The action is still invariant under the supersymmetry transformations (5). The field equations are satisfied automatically by $\psi_\alpha = 0$ and time independent $\phi(x)$ which satisfy

$$\left(1 - \frac{\partial_x^2}{M^2}\right)\partial_x\phi \mp W'(\phi) = 0 \tag{18}$$

One can observe directly that this equation is really needed to minimize potential energy of the system. Just multiply left hand side of (18) by

$$\partial_x\phi \mp \left(1 - \frac{\partial_x^2}{M^2}\right)^{-1} W'(\phi)$$

and the result is equal to a potential energy term of Lagrangian up to total derivatives.

The nontrivial solution of (18) with $-$ sign we call supersymmetric kink with higher-derivative regularization.

Now we compute the supersymmetric kink shape up to the first order in $1/M^2$. The field equation

$$\left(1 - \frac{\partial_x^2}{M^2}\right)\partial_x^2\phi - W''\left(\frac{1}{1 - \frac{\partial_x^2}{M^2}}W'\right) = 0$$

has the nontrivial solution of the form

$$\phi(x) = \frac{m}{2\lambda}\tanh\frac{mx}{2} + \frac{1}{M^2}f(x) + O(\frac{1}{M^4})$$

after change of variables $\zeta = \tanh\frac{mx}{2}$ we arrive at the following equation for $f(\zeta)$

$$\frac{\partial}{\partial\zeta}(1-\zeta^2)\frac{\partial}{\partial\zeta}f(\zeta) + 6f(\zeta) - \frac{4}{1-\zeta^2}f(\zeta) + \frac{m^3}{2\lambda}(9\zeta^3 - 5\zeta) = 0$$

which has the exact solution of the form

$$f(\zeta)=\frac{1}{\zeta^2-1}\left[\frac{m^3}{12\lambda}\zeta(19-24\zeta^2+9\zeta^4)-c_2(10\zeta-6\zeta^3)\right]+c_1(\zeta^2-1)+3c_2(1-\zeta^2)\ln\frac{1+\zeta}{1-\zeta}$$

with arbitrary constants c_1 and c_2. We fix $c_2=\frac{m^3}{12\lambda}$ in order for our solution not to have singularities at $\zeta=\pm 1$ (i.e. $x=\pm\infty$). So, the result for $f(x)$ is

$$f(x)=-\cosh^{-2}\frac{mx}{2}\left(\frac{3m^3}{4\lambda}\tanh\frac{mx}{2}+c_1+\frac{m^4}{4\lambda}x\right)$$

The solution $\phi(x)$ satisfies our kink condition (18) for any c_1. It is easy to check that it does not change the classical kink energy (of course, this is true up to $\frac{1}{M^4}$ order only). This indicates that c_1 is a zero mode for our kink. It is really the first order in $\frac{1}{M^2}$ contribution to the translational zero mode.

In the rest of this paper we will be computing a lot of vacuum expectation values. This will be done using the propagators in trivial sector which read [2]

$$\frac{M^2}{p_x^2+M^2}\frac{1}{p^2-W''(\phi_0)^2\left(1+\frac{p_x^2}{M^2}\right)^{-2}}$$

for bosons and

$$\frac{M^2}{p_x^2+M^2}\frac{\not{p}+W''(\phi_0)\left(1+\frac{p_x^2}{M^2}\right)^{-1}}{p^2-W''(\phi_0)^2\left(1+\frac{p_x^2}{M^2}\right)^{-2}}$$

for fermions. Actually, the only thing which is important to us is the ultraviolet behavior of these propagators. As in the momentum cut-off scheme, the great difference with the computation in trivial sector will come because our classical bosonic background is now an odd function of coordinate.

3.1 Supercurrent

To obtain the supersymmetry current we extend Noether method for the case of higher-derivatives. First, we integrate our Lagrangian by parts to rewrite it in the form with at most two derivatives on each field.

Consider some general Lagrangian $L(\phi,\partial_\mu\phi,\partial_\mu\partial_\nu\phi)$. The Lagrange equation has the form

$$\frac{\delta L}{\delta\phi}-\partial_\mu\frac{\delta L}{\delta(\partial_\mu\phi)}+\partial_\mu\partial_\nu\frac{\delta L}{\delta(\partial_\mu\partial_\nu\phi)}=0$$

For local parameter $\epsilon=\epsilon(t,z)$ we write the variation of the $L(\phi,\partial_\mu\phi,\partial_\mu\partial_\nu\phi)$ as

$$\delta L=\partial_\mu k^\mu+(\partial_\mu\bar{\epsilon})j^\mu \tag{19}$$

where $k^\mu = \bar{\epsilon} k_0^\mu + (\partial_\nu \bar{\epsilon}) k_1^{\nu\mu}$ One may check directly that the current, defined this way, is conserved on shell. To do this, we note that

$$\delta L = \frac{\delta L}{\delta \phi}\delta\phi + \frac{\delta L}{\delta(\partial_\mu \phi)}\delta(\partial_\mu\phi) + \frac{\delta L}{\delta(\partial_\mu\partial_\nu\phi)}\delta(\partial_\mu\partial_\nu\phi) \tag{20}$$

where we write $\delta\phi = \bar{\epsilon}\Delta\phi$. Now, plug k^μ into (19) and equate terms with equal number of derivatives on $\bar{\epsilon}$ in (19) and (20). The result is

$$\partial_\mu k_0^\mu = \frac{\delta L}{\delta \phi}\Delta\phi + \frac{\delta L}{\delta(\partial_\mu\phi)}(\partial_\mu \Delta\phi) + \frac{\delta L}{\delta(\partial_\mu\partial_\nu\phi)}(\partial_\mu\partial_\nu\Delta\phi)$$

$$k_0^\mu + j^\mu + \partial_\nu k_1^{\mu\nu} = \frac{\delta L}{\delta(\partial_\mu\phi)}\Delta\phi + 2\frac{\delta L}{\delta(\partial_\mu\partial_\nu\phi)}\partial_\mu\Delta\phi$$

$$k_1^{\nu\mu} = \frac{\delta L}{\delta(\partial_\mu\partial_\nu\phi)}\Delta\phi$$

Taking ∂_μ of second equation, and using the remaining two to express derivatives of k_0^μ and $k_1^{\nu\mu}$, we obtain on shell $\partial_\mu j^\mu = 0$, the current is conserved.

The current is not unique though. Using this method one can compute, for example, any of the following currents:

$$j^\mu_{PvN} = (\partial_\nu\phi)\gamma^\nu\gamma^\mu\left(1 - \frac{\partial_x^2}{M^2}\right)\psi + iW'\gamma^\mu\psi + \frac{\delta_x^\mu}{M^2}\left[(\Box\phi)\overleftrightarrow{\partial_x}\psi + iF\overleftrightarrow{\partial_x}(\not\partial\psi)\right] \tag{21}$$

$$j^\mu_{SVV} = j^\mu_{PvN} - \frac{1}{M^2}\epsilon^{\mu\nu}\partial_\nu(\not\partial\phi\overleftrightarrow{\partial_x}\gamma^0\psi) \tag{22}$$

$$j^\mu_{OK} = j^\mu_{PvN} + \frac{1}{M^2}\epsilon^{\mu\gamma}\partial_\gamma\left[(\partial_\nu\phi)\gamma^\nu\gamma^0(\partial_x\psi)\right] \tag{23}$$

Clearly, the conservation of any of these current implies the conservation of the others. The confusing issue for the last year was in that if we use methods of [2] for each of these currents we compute different result for the anomaly. Below we will solve this problem by analizing the contributions from the bosonic commutators. At the same time, we will describe that only one of these three currents, j^μ_{OK} is regulated properly for loop computations. Nevertheless, we will formally compute the anomaly values from all three currents and observe that these values coincide.

For the zero components of these currents we write [4]

$$j^0_{PvN} = (\partial_\nu\phi)\gamma^\nu\gamma^0\left(1 - \frac{\partial_z^2}{M^2}\right)\psi + iW'\gamma^0\psi \tag{24}$$

[4] Actually, it is easy to guess the j^0_{OK} without doing any computations: by the integration by parts the Lagrangian can be put in the form of sum of nonregulated lagrangian, which gives first two terms in j^0_{OK}, and kinetic part of Lagrangian for fields $\partial_x\phi$, $\partial_x\psi$ and $\partial_x F$. The susy transformations for that field differ from susy transformations of prime fields by the term $\partial_x\epsilon$, which would contribute only to the first component of the supercurrent, so one may write the "free" supercurrent for this derivative fields in j^0_{OK} , this gives the third term.

$$j^0_{SVV} = \left(\partial_\nu \left(1 - \frac{\partial_z^2}{M^2}\right)\phi\right)\gamma^\nu\gamma^0\psi + iW'\gamma^0\psi \tag{25}$$

$$j^0_{OK} = (\partial_\nu\phi)\gamma^\nu\gamma^0\psi + iW'\gamma^0\psi + \frac{1}{M^2}(\partial_\nu\partial_x\phi)\gamma^\nu\gamma^0\partial_x\psi \tag{26}$$

3.2 Supercharge Algebra and Central Charge VEV

In this section we will compute central charge part of the supersymmetry algebra. We define topological current ζ via

$$\{j^\mu_\alpha, \bar{Q^\beta}\} = 2(\gamma_\nu)^\beta_\alpha T^{\mu\nu} + 2i(\gamma^5)^\beta_\alpha \zeta^\mu$$

and the central charge is

$$Z = \int dz \zeta^0$$

so that supercharges anticommute as (10). Note, that expression (10) does not contain δ^β_α term due to symmetry. The corresponding term could still be present in current charge anticommutator if it's zero component space integral is zero.

There are actually two equivalent ways to compute central charge. One is to use the Poisson brackets directly

$$\left[\phi(t,x), \left(1 - \frac{\partial^2_{x'}}{M^2}\right)\dot{\phi}(t,x')\right] = i\delta(x - x') \tag{27}$$

$$\left\{\psi_\alpha(t,x), \left(1 - \frac{\partial^2_{x'}}{M^2}\right)\psi_\beta(t,x')\right\} = \delta_{\alpha\beta}\delta(x - x'). \tag{28}$$

Another way is to note that $\delta j^\mu = [\bar{Q}\epsilon, j^\mu]$ and to use the supersymmetry transformations for fields. Before doing so, one should make sure that the charge Q does really generate proper transformations of ψ and ϕ, which again requires the use of commutation relations.

Let us look at equation (28). The right hand side is an even function of $x - x'$. Clearly, the left hand side should also be symmetric with respect of exchanging of x and x'. From this we conclude that we can change ∂_x by $\partial_{x'}$ in our operator. Is it very unusual for quantum commutation relations to have nonlocal behavior? We remind that exactly the same phenomenon occur in the sharp momentum cut-off method of previous section. Possibly this is a general property of any regularization scheme which deserves future study.

We are going to work out the bracket for all three currents, but we want to start with j^μ_{SVV} because it was addressed by original paper [2]. The central charge term in the anticommutator for j^0_{SVV} has again two parts, corresponding to cases 1) and 2) of the momentum cut-off regularization computation, presented in the first chapter. The case 1) part comes from anticommutator of the fermionic fields multiplied with bosonic loop, and has the form

$$2Z_{SVV,bos} = \int dx \int dx' \left(\left(1 - \frac{\partial_x^2}{M^2}\right)\partial_x\phi(x)\right) W'(\phi(x'))\left(1 - \frac{\partial^2_{x'}}{M^2}\right)^{-1}\delta(x - x') +$$

$$\int dx \int dx' \left(\left(1 - \frac{\partial_{x'}^2}{M^2}\right) \partial_{x'}\phi(x') \right) W'(\phi(x)) \left(1 - \frac{\partial_{x'}^2}{M^2}\right)^{-1} \delta(x - x') \tag{29}$$

At this point we would like to give an example. Consider for a second $\int dy\, (f(y)\partial_y \delta(x-y))$ There are two formal ways to compute this integral. One is integration by parts which gives $f(y)\delta(x-y)|_{boundary} - \partial_x f(x)$. Another way is to put $\partial_y \delta(x-y) = -\partial_x \delta(x-y)$ and switch the order of the integral and the derivative. If x is away from an integration interval, both give the same answer. If not, the second way is wrong because $\int dy f(y)\delta(x-y)$ is not continuous function of x, it has a jump when x crosses the boundary of the integration region. If we would use the second method for (29), we would get [2]

$$Z_{SVV,bos} = \frac{1}{2}\left[W'\partial_x\phi + \left(\left(1 - \frac{\partial_x^2}{M^2}\right) \partial_x\phi \right) \left(\left(1 - \frac{\partial_x^2}{M^2}\right)^{-1} W' \right) \right]$$

We consider this intermediate result of [2] as being wrong.

On the other hand, just notice that we can change x' by x in the last $\left(1 - \frac{\partial_{x'}^2}{M^2}\right)^{-1}$, and then both terms become equal. I.e.,

$$2Z_{SVV,bos} = 2\int dx \int dx' \left(\left(1 - \frac{\partial_x^2}{M^2}\right) \partial_x\phi(x) \right) W'(\phi(x')) \left(1 - \frac{\partial_{x'}^2}{M^2}\right)^{-1} \delta(x - x')$$

(Note, that if we would apply the "second method" to this result we would get the classical answer, without any anomaly at all!).

Here we need to define what we mean by $\left(1 - \frac{\partial_{x'}^2}{M^2}\right)^{-1} \delta(x - x')$ and by $\delta(x - x')$ itself. If we had imposed boundary conditions for the quantum fields, we would need to redefine Dirac delta function as a sum of the modes with corresponding boundary conditions, as described in [4]. Nevertheless, we try to avoid imposing boundary conditions here, following the ideas of [2]. So, we can just use $\delta(x - x') = \int \frac{dq}{2\pi} \exp(iq(x - x'))$ and then the regularization procedure spreads it to

$$\hat{\delta}(x - x') = \left(1 - \frac{\partial_{x'}^2}{M^2}\right)^{-1} \delta(x - x') = \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \frac{\exp(iq(x - x'))}{1 + \frac{q^2}{M^2}} = \frac{M}{2} \exp(-M|x - x'|) \tag{30}$$

Now we can rewrite the VEV of $Z_{SVV,bos}$ in the form, analogous to (12), and the anomaly is given by the term

$$\int_{-L}^{L} dx \int_{-L}^{L} dx' W''(x) \langle \eta(x) \partial_{x'} \left(1 - \frac{\partial_{x'}^2}{M^2}\right) \eta(x') \rangle \hat{\delta}(x - x') \tag{31}$$

Actually, the effect of regularization on the propagator is removed by the factor $\left(1 - \frac{\partial_{x'}^2}{M^2}\right)$ in the correlator. Effectively, we get nonregularized propagator. The loop integral is then divergent at all and there is no fare way to compute such a thing. In the same time, one is tempted to cheat: let us think that the loop integral runs up to some finite value of

momentum, and then we may first take x integrals, and after that put that momentum to infinity. If we do that, it shouldn't surprise the reader that we get the result which we argued to be correct in section 1. (In this case the fermionic contribution is zero.) We just have the old situation with under-regulated propagator, with exactly the same consequences. Thus, though the current j^μ_{SVV} is not regulated properly by our scheme and that is why it should not be used for further computations, the result that it gives is formally correct.

Now we move to the current j^μ_{PvN}. The computation with this current is easy, and the resulting integral for an anomaly is

$$\int_{-L}^{L} dx \int_{-L}^{L} dx' W''(x)\langle\eta(x)\partial_{x'}\eta(x')\rangle\delta(x-x') \tag{32}$$

which gives well regulated propagator, but, unfortunately, nonregularized delta function. Here, of course, the result for an anomaly coming from the bosonic sector is zero (as it has to be in the case where delta function is under-regulated). But the fermionic sector contributes now and the total anomaly is again the correct one. Thus, though the current j^μ_{PvN} does not supply well regulated expression either, the formal result for the anomaly is the same.

Now we move to the winning one, the current j^μ_{OK}. The first two terms in it give the well regulated expression

$$\int_{-L}^{L} dx \int_{-L}^{L} dx' W''(x)\langle\eta(x)\partial_{x'}\eta(x')\rangle\hat{\delta}(x-x') \tag{33}$$

Using our trick for the substitution of kink shape by a step shape (see Appendix A) , we arrive at the following expression for this integral

$$W''(L)\frac{iM}{2}\int_{-L}^{L} dx \;\; \mathrm{sign}(x)\int_{-L}^{L} dx' \int \frac{d^2p}{(2\pi)^2}\frac{-ip_x}{\left(1+\frac{p_x^2}{M^2}\right)p^2}\exp(-M|x-x'|+ip(x-x')) \tag{34}$$

which is easy to compute (see Appendix B) the result for $L\to\infty$ being

$$\frac{W''(\infty)}{4\pi} \tag{35}$$

which is the half of the anomaly. Clearly, another half is coming from the fermionic loop, which correspond to case 2) of the first chapter. The resulting integral is exactly equal to (33), so that the total anomaly is again

$$\frac{W''(\infty)}{2\pi}$$

It remains to analyze the last term in the j^μ_{OK}. This term gives the following contribution to the central charge anomaly

$$W''(L)\frac{1}{M^2}\int_{-L}^{L} dx \;\; \mathrm{sign}(x)\int_{-L}^{L} dx' \int \frac{d^2p}{(2\pi)^2}\frac{p_x^2\exp(ip_x(x-x'))}{p^2\left(1+\frac{p_x^2}{M^2}\right)}\int\frac{dq}{2\pi}\frac{q\exp(iq(x-x'))}{\left(1+\frac{p_x^2}{M^2}\right)}$$

An easy computation shows that this integral is zero (for $x \neq x'$, both momentum integrals converge and we can change the order of the integration, the result is zero. For $x = x'$, change in the order of the integration is not needed as the last momentum integral is seen to be zero on its own).

This concludes our treatment of an anomaly in the higher-derivative regularization.

We would like also to comment on the supercharges which follow from the currents which we have discussed. If one writes the supercharges in components, following [9], then the only one supercharge, that looks as though it would have zero second component, is Q_{SVV}, namely

$$Q_{SVV,1} = \int dz \left[\left(\left(1 - \frac{\partial_z^2}{M^2} \right) \partial_t \phi \right) \psi_1 + \left(\left(1 - \frac{\partial_z^2}{M^2} \right) \partial_z \phi \right) \psi_2 + W' \psi_2 \right]$$

$$Q_{SVV,2} = \int dz \left[\left(\left(1 - \frac{\partial_z^2}{M^2} \right) \partial_t \phi \right) \psi_2 + \left(\left(1 - \frac{\partial_z^2}{M^2} \right) \partial_z \phi \right) \psi_1 - W' \psi_1 \right]$$

and ψ_1 term in $Q_{SVV,2}$ reproduces the Bogomol'nyi equation (18). Nevertheless, the quantum loops with this current are not regulated, as we have described above.

Finally, we would like to comment on another approach to the anomaly computation based on the dimensional regularization scheme (DREG) and the one given in [2]. We are confident that if we could repeat our momentum cut-off program here, we would then get the same value for the anomaly. Nevertheless, after reading our paper and the paper [2] one may be slightly confused: is the anomaly from momentum cut-off and higher-derivative scheme, presented above, the same as the dimensional regularization anomaly of [2]? Our answer is "no". We feel that the anomaly in DREG should arise exactly the same way as it does in other schemes, from the same integral.

The computation of [2] says that

$$(\gamma^\mu J_\mu) = (D-2)(\partial_\mu \phi)\gamma^\mu \psi - DiW'\psi$$

and using

$$j^0 = \partial_\mu \gamma^\mu \gamma^0 \psi + iW' \gamma^0 \psi$$

one easily computes

$$\left\{ (\gamma^\mu J_\mu)_{anom}(z), \bar{j}^0(z') \right\} = (D-2)\partial_\mu \phi(z) \gamma^\mu \gamma^\nu (\partial_\nu \phi) \hat{\delta}(z-z') +$$

$$iW'(D-2)(\partial_\mu \phi)\gamma^\mu \hat{\delta}(z-z') - iDW'(\partial_\nu \phi)\gamma^\nu \hat{\delta}(z-z')$$

it is only at this point when one must take quantum average, and we observe that $D-2$ with $-D$ give just 2. No anomaly is seen on this level!

The detailed treatment of the dimensional regularization scheme was given by [6]. The recent paper [10] has proven that the counterterm, used by [6] is really equal to the tadpole graph contribution, and thus removes tadpoles completely. This is the specific property of dimensional regularization. If one thinks of the results of [6] in the context of another regularization scheme, the counterterm needed may be different. For example, if one thinks of

the momentum cut-off scheme, the traditional counterterm of [3] and [4] differs from the one used in [6] by the integration by parts, i.e. by the finite boundary term. Nevertheless, this two counteterms seem to coincide when dimensionally regulated. We find this miraculous.

We stress that the results of [6] are only the dimensional regularization results. It is well known, that the naive momentum cut-off gives wrong results for mass, the reason for that we have described in [1]. On the same time, it is the momentum cut-off that is being implicitly assumed in the mass computation of [6] when the authors unite different sums/integrals over the continuous spectrum under one integral sign replacing the densities of states by derivative of phase shift. It is easy to check that the usage of the wrong counterterm brings in exactly that finite correction to the mass, that was lost when doing the momentum cut-off. So the final result looks to be correct. This may be not the coincidence and there still remains something to be studied.

We are also surprised by the way which [6] uses to compute the one loop correction for the central charge. This method does not disclose the anomalous nature of the correction, and it sounds to us as the computation of $\langle Z\rangle = \langle H\rangle + \langle Q_2^2\rangle$, i.e. we feel that what is really proved is that $\langle Q_2^2\rangle$ is zero. This and other issues about the phase-shift method will be discussed elsewhere [1].

4 Conclusions

In this paper we have developed a new approach to the momentum cut-off regularization, which we call *generalized* momentum cut-off regularization scheme. This new technique resolves the issue of the one loop anomaly in the central charge of the supersymmetric kink with a sharp momentum cut-off regularization. We have imposed a cut-off not only for the loop integrals (this cut-off we call Λ), but also for the Dirac delta functions themselves (the cut-off that we call K). The consistency required that these two cut-offs are equal. We emphasize that if they were not, such a theory would be internally inconsistent. In particular, if $\Lambda > K$ then such a theory contains "purely classical" modes in operator expressions, i.e. the modes multiplied by perfectly commuting (anticommuting) creation/absorption "operators", and this gives a different result for an anomaly. On the other hand, if $\Lambda < K$ then the commutators for quantum fields are inconsistent, because then for fields $\phi(x) \sim \exp(iqx)$ $K > q > \Lambda$ the right hand side of commutation relations can be nonzero and the left hand side can not.

Note that imposing momentum cut-off on the Dirac delta functions makes them nonlocal. The same effect happens in the higher-derivative regularization scheme, and may be possibly a common feature of regularized quantum field theories. It is not immediately clear to us that this nonlocality follows from the nonlocality of quantum propagators, which is, of course, the meaning of regularization. The direct consequence of this is that the (anti)commutators of different symmetry currents result in currents that are not local but spread over the finite interval of the size $\sim 1/\Lambda$. In particular, the topological current that results from the susy algebra is not local. Again, this is quite natural once we start from the very beginning with the model where there are no local objects – of course, one can not think about perfectly localized quantum field if only the Fourier harmonics up to Λ are allowed.

We have also elaborated on the higher-derivative regularization scheme of [2], as used for the supesymmetric kink. We gave consistent technical prescriptions for this method and we have computed the correct value of an anomaly in this scheme. The reason that we dwell on this topic is that the scheme is preserving supersymmetry manifestly, and we find it useful for further investigations of the higher-dimensional problems.

In the higher-derivative regularization scheme we notice that the susy current is not unique – one may change it by adding a full derivative term. Not all the currents produced this way are regulated by the scheme, so one should pick only a very special current, namely one for which one computes from the susy algebra a regularized topological current, i.e. the topological current such that when one takes the vacuum expectation value of it, the regularization effect in propagator is not removed by extra powers of momentum in the vertex. The topological current that we obtain this way is not local. The nonlocality is the reason that the anomaly is present. We conclude that it is essential that the topological currents are not local in any regularization scheme, otherwise no anomaly would be seen. It would be definitely very instructive to see how does this nonlocality work in the case of the dimensional regularization, but in this paper we only comment on the earlier approaches to the dimensional regularization and argue that in our opinion the problem is still not solved.

We have checked that the properly regulated supersymmetric kink system has the central charge anomaly which is exactly equal to the one loop mass correction computed by [4],[6],[7]. This concludes the discussion on BPS bound saturation. We have worked out in great detail two regularization schemes, but the conclusion looks to be quite general: There are only three results possible. If one gets the zero or twice the proper anomaly results, this means that the theory does not regulate properly the loops for the superalgebra. The anomaly always comes from the nonlocal topological currents due to the integration in the vicinity of the boundary. I.e. in the properly regulated theory the boundary itself is smeared over the finite interval. The length of this interval is defined by the inverse of the regulator. Nothing can be left local in the theory which is regulated properly, neither the position of the boundary, nor the delta functions in the canonical commutation relations.

Acknowledgments: Through all my work, I have always felt greatest support of my parents, Vitalii and Natalia. This work would be impossible without ideas, help and encouraging by my teacher and advisor, Peter van Nieuwenhuizen. I benefited a lot from the collaboration with Alfred Goldhaber and Misha Stephanov, and from the discussions with Bogdan Kulik. I am very grateful to M.Shifman, A.Vainshtein and M.Voloshin for very useful critical discussion of some of the results presented here. It is a pleasure to acknowledge the discussions and correspondence with E.Farhi, N.Graham, R.L.Jaffe, A.Rebhan, C.Vafa and H.Weigel. I am very grateful to Liliya Vugmeister for proof-reading the draft of this paper.

A Computation of the integral for the anomalies in the momentum cut-off regularization

The computation of the integral (13) is easy. Notice, that if we would replace the kink background with the step like shape: $W''(x) \to W''(L)\mathrm{sign}(x)$, then we would compute the same answer for this integral. There is good reason for that: if we put $L = \infty$ from the very start, we see that this integral is just zero. I.e., the integral in the limits $(-L, L)$ is equal to the negative of the same integral over the region $|x| > L$, and for sufficiently large L such substitution is justified. We will be using the same trick when working with the higher-derivative regularization integrals.

We start with the p_0 integral in (13). Remember that we put momentum cut-off on p_x integral, while we integrate in p_0 over all real axes. The integral can be performed with residues or by Wick rotation, either way gives

$$\frac{i}{8\pi^3} 2\pi i \frac{i}{2} W''(L) \int_{-L}^{L} dx \int_{-L}^{L} dx' \int_{-\Lambda}^{\Lambda} dp \int_{-K}^{K} \mathrm{sign}(x)\mathrm{sign}(p) \exp(i(x-x')(p+q) \tag{36}$$

(we drop subindex x of p_x). Next, we take the coordinate integrals to obtain

$$-\frac{i}{8\pi^2} W''(L) \int_{-\Lambda..\Lambda} dp \int_{-K}^{K} dq 2i \mathrm{sign}(p) \frac{2\sin(L(p+q)) - \sin(2L(p+q))}{(p+q)^2} \tag{37}$$

Integral over p from $-\Lambda$ to Λ is clearly twice the integral from 0 to Λ. We introduce $z = p+q$ and get

$$-\frac{i}{8\pi^2} W''(L) 2 \int_{0}^{\Lambda} dp \int_{p-K}^{p+K} dz 2i \frac{2\sin(Lz) - \sin(2Lz))}{z^2} \tag{38}$$

next, we do z integral which yields

$$\frac{1}{2\pi^2} W''(L) \int_{0}^{\Lambda} dp \tag{39}$$

$$\Bigg(\frac{(-4\sin(Lp)\cos(LK) + 2\sin(2Lp)\cos(2LK))K}{K^2 - p^2}$$

$$+ \frac{(4\cos(Lp)\sin(LK) - 2\cos(2Lp)\sin(2LK))p}{K^2 - p^2}$$

$$+ L(\mathrm{Ci}((p+K)L) - \mathrm{Ci}(2(p+K)L) - \mathrm{Ci}((p-K)L) + \mathrm{Ci}(2(p-K)L)))$$

where the function $\mathrm{Ci}(x) = \gamma + \ln(x) + \int_0^x \frac{\cos t - 1}{t} dt$ is integral cosine function. If $K > \Lambda$ then no terms in this expression are singular. One may compute the expressions, but it is clear already now that they will come in pairs such that for any term with L there is a counterpart with $2L$, so taking $L \to \infty$ limit will cancel all the terms and the answer is zero. The situation is different for $K \leq \Lambda$. The last line in (39) is still zero - it is clear from the definition of Ci(x): The singularities in Ci are logarithmic (using that $\int \ln x dx = x \ln x - x + const$, we see that each of them separately would give finite contribution) and for the last line of (39) they are seen to cancel for any given L. At the same time, the infrared singularity develops

in the fractions, the first two terms of (39). We can not argue about the value of the integral of these fractions until we take it explicitly. This is easy to do, the result being

$$\frac{1}{2\pi^2}W''(L)\left(\mathrm{Si}(2\Lambda L+2KL)-2\mathrm{Si}(LK-\Lambda L)-\right.$$

$$\left.2\mathrm{Si}(\Lambda L+KL)+\mathrm{Si}(-2\Lambda L+2KL)-2\mathrm{Si}(2KL)+4\mathrm{Si}(KL)\right)$$

with $\mathrm{Si}=\int_0^x \frac{\sin t}{t}$ being the integral sine. The limit $L=\infty$ is then trivial and our final result is

$$\frac{W''(\infty)}{4\pi}\left(1+\mathrm{sign}(\Lambda-K)\right) \tag{40}$$

which is exactly the result (14).

B Computation of the integral for the anomalies in the higher-derivatives regularization

Here we compute (34). This computation is much easier than one in the momentum cut-off scheme. First, take p_0 integral, then x and x', and finally p_x integrals to get

$$-\frac{W''(L)}{2\pi}\left(\exp(3ML)(8-4ML)-4\exp(4ML)+\right.$$

$$\mathrm{Ei}(1,ML)\exp(4ML)(2-4ML+4(ML)^2)+\mathrm{Ei}(1,2ML)\exp(4ML)(-1+4ML-8(ML)^2)+$$

$$\left.4\exp(2ML)(ML-1)+\mathrm{Ei}(1,-2ML)-2\mathrm{Ei}(1,-ML)\exp(2ML)\right)\exp(-4ML)$$

where $\mathrm{Ei}(n,x)=\int_1^\infty \frac{\exp(-xt)}{t^n}dt$ is integral exponent function. Taking the limit $L\to\infty$ gives the result (35).

Notice that to compute this result we have substituted the exact kink shape by the sign function. As we have described already, technically the reason is that the nonzero value of this integral comes from the regions around the boundary. This means that the boundary is not localized at one point. It is smeared over the interval of the length $1/M$. Nothing local can be left in the properly regulated theory!

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